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Tunnelling in supersymmetric QCD

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Abstract. Vacuum structure in SQCD is analysed. It is pointed out, in particular, that quantum mechanical tunnelling in the massive theory is mediated by pure instantons and zero classical scalar fields due to the trivial topological structure of the set of minima of the classical potential. In the massless theory, on the other hand, the set of minima has a non-trivial topology, as a result of which an instanton has to be accompanied by a topologically non-trivial scalar field in order to mediate the tunnelling. Indeed, this configuration has an infinite action if a mass term is added, and its tunnelling is completely suppressed unlike that of the trivial scalar configuration.

Supersymmetry breaking was extensively studied recently [1-3]. Since supersymmetry cannot be broken perturbatively it is the non-perturbative effects, mainly instantons, which were used to analyse possible effects of supersymmetry breaking. Whereas in pure Yang-Mills theory vacuum structure is known, it is less known in theories which include scalar fields, such as supersymmetric QCD. This is, of course, important if tunnelling effects are to be considered.

In pure Yang-Mills theory the vacuum is labelled by an index n, the Pontryagin index. When scalar fields are added the topology of the set of minima of the classical potential has to be taken into account. Normally if the gauge group is G broken down to H, this set of minima, M, can be taken as the coset space G/H‡. Thus if it has a non-trivial topological structure, there is another index, k, which characterises the vacuum.

In the following we analyse the vacuum structure in SQCD with gauge group SU(2) and show that $k = n$ (or $k = 0$) if the scalar fields are in the fundamental representation and the theory is massless, and $k = 0$ in the massive theory. That means, in particular, that in the massless theory tunnelling amplitude between $|n\rangle \rightarrow |n+1\rangle$ goes through Euclidean configurations which are topologically non-trivial for the gauge potentials (instantons) and the scalar fields simultaneously, whereas in the massive theory the instanton is accompanied by a classical scalar configuration, $\phi = 0$. Indeed, if the topologically non-trivial scalar configuration of the massless theory is used in the massive theory, tunnelling is totally suppressed because the classical action diverges due to the contribution of the mass term.

The Lagrangian of an SU(2) supersymmetric model contains, apart from the gauge supermultiplet, one matter and one antimatter supermultiplets transforming under the fundamental representation of the gauge group. It is given in the massless theory by

$$\mathscr{L} = \mathscr{L}_{\mathrm{SYM}} + \mathscr{L}_{\mathrm{matter}} \tag{1}$$

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‡ In theories with an accidental symmetry M may not coincide with G/H. But such a structure does not generally survive quantum corrections, as a result of which M can be identified in the quantum level with G/H.

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where $\mathscr{L}_{\mathrm{SYM}}$ is the super-Yang-Mills Lagrangian given in the Wess-Zumino gauge by

$$\mathscr{L}_{\mathrm{SYM}} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \bar{\lambda}^a \mathrm{i} D_\mu \bar{\Sigma}^\mu \lambda^a + \mathrm{HC} \tag{2}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu$, $D^{ac}_\mu = \delta^{ac} + g\varepsilon^{abc}A^b_\mu$, A^a_μ are the gauge potentials and λ^a are Weyl spinors. They are expressed in Weyl basis with the Dirac matrices being

$$\gamma_\mu = \begin{pmatrix} 0 & \Sigma_\mu \\ \bar{\Sigma}_\mu & 0 \end{pmatrix} \qquad \Sigma_\mu = (\mathbb{1}, \sigma_i) \qquad \bar{\Sigma}_\mu = (\mathbb{1}, -\sigma_i)$$

and $\mathrm{Tr}(\Sigma_\mu \bar{\Sigma}_\nu) = 2g_{\mu\nu}$ with Minkowskian metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. $\mathscr{L}_{\mathrm{matter}}$ is the matter field Lagrangian

$$\begin{aligned}\mathscr{L}_{\mathrm{matter}} = {} & (\mathscr{D}_\mu\phi_1)^\dagger \mathscr{D}^\mu\phi_1 + (\mathscr{D}^*_\mu\phi_2)^\dagger \mathscr{D}^{\mu *}\phi_2 + \bar{\psi}_1^{\mathrm{T}} \mathrm{i}\mathscr{D}_\mu \bar{\Sigma}^\mu \psi_1 + \psi_2^{\mathrm{T}} \mathrm{i}\mathscr{D}^*_\mu \Sigma^\mu \bar{\psi}_2 \\ & + \frac{\mathrm{i}g}{\sqrt{2}}(\phi_1^\dagger \tau^a \lambda^a \psi_1 + \phi_2^{\mathrm{T}} \tau^a \bar{\lambda}^a \bar{\psi}_2) - \tfrac{1}{32}g^2(\phi_1^\dagger \tau^a \phi_1 - \phi_2^{\mathrm{T}} \tau^a \phi_2^*)^2 + \mathrm{HC}\end{aligned} \tag{3}$$

such that $\{\psi_i, \phi_i\}$ $(i = 1, 2)$ form the matter supermultiplets, $\mathscr{D}_\mu = \partial_\mu + \mathrm{i}gA^a_\mu \tfrac{1}{2}\tau^a$ and $\tfrac{1}{2}\tau^a$ are the SU(2) generators in the fundamental representation.

Classically the vacuum is given by the zero-energy states. These are defined by $\psi_1 = \psi_2 = \lambda = F_{\mu\nu} = 0$, ϕ_1 parallel to ϕ_2^* in group space and $\mathscr{D}_\mu\phi_1 = \mathscr{D}^*_\mu\phi_2 = 0$. The first condition defines the vector potentials to be pure gauge, $A^a_\mu \tfrac{1}{2}\tau^a = (\mathrm{i}/g)U^{-1}(\boldsymbol{x})\partial_\mu U(\boldsymbol{x})$ where $U(\boldsymbol{x}) \in \mathrm{SU}(2)$ and $\boldsymbol{x} \in E_3$ (Euclidean 3-space). These are the gauge transformations left after fixing the gauge $A^a_0 = 0$. We compactify E_3 into S_3 by imposing the boundary conditions $U(\boldsymbol{x}) \rightarrow_{|\boldsymbol{x}| \to \infty} \mathbb{1}$. Then $U(\boldsymbol{x})$ belongs to the set of maps $S_3 \to \mathrm{SU}(2)$ classified by the third homotopy group of SU(2), $\pi_3(\mathrm{SU}(2)) = \mathbb{Z}$.

The second requirement $\mathscr{D}_\mu\phi_1 = 0$, has an integrability condition $F^a_{\mu\nu}\tfrac{1}{2}\tau^a\phi_1 = 0$, which is trivially satisfied in the vacuum $(F^a_{\mu\nu} = 0)$. The solution is then

$$\phi_1 = \phi_2^* = P \exp\left(-\mathrm{i}g \int_\Gamma A^a_\mu \frac{\tau^a}{2} \mathrm{d}y^\mu\right) v$$

where the integral is along a path Γ from $-\infty$ up to $\boldsymbol{x}$ and v is a constant vector in the fundamental representation of the group. The integrability condition guarantees that ϕ_i $(i = 1, 2)$ is independent of the path. It is a function of x only. Indeed, by substituting A^a_μ, which is a pure gauge, it is easily found that $\phi_1(\boldsymbol{x}) = \phi_2^*(\boldsymbol{x}) = U(\boldsymbol{x})v$. Thus $\phi_i(\boldsymbol{x})(i = 1, 2)$ define an S_3 of arbitrary radius $|v|$, including zero. For $v \neq 0$, $\phi_i(\boldsymbol{x})$ like $U(\boldsymbol{x})$ belong to the set of maps $\phi_i : S_3 \to S_3$ classified by $\pi_3(S_3) = \mathbb{Z}$†. With this we find that the configuration space of the scalar fields and the gauge potentials are classified by the same index n, the Pontryagin index. In particular they are both topologically non-trivial simultaneously. When $v = 0$, on the other hand, there exists a special set of vacua with $\phi_1 = \phi_2 = 0$. The scalars have a trivial topology $(k = 0)$, whereas the Yang-Mills potentials are classified by the Pontryagin index n as in the massive theory (this will be shown later).

The vacuum is then given by $|\theta\rangle = \sum_{n=-\infty}^{\infty} \exp(\mathrm{i}n\theta)|n\rangle$ and quantum mechanical tunnelling between vacua differing by one unit of topological charge is provided by the single instanton or single anti-instanton:

$$A^{Ia}_\mu = \frac{2}{g}\frac{\eta_{a\mu\nu}(x - x_1)_\nu}{(x - x_1)^2 + \rho_1^2} \qquad A^{\bar{I}a}_\mu = \frac{2}{g}\frac{\eta_{a\mu\nu}(x - x_2)_\nu}{(x - x_2)^2 + \rho_2^2} \tag{4}$$

† Generally v_i, the length of ϕ_i $(i = 1, 2)$, is not a constant and the scalar fields do not define an S_3. However, for the vacuum v = constant and we get the maps $S_3 \to S_3$.

and the scalar field configurations [4] (when $v \neq 0$)

$$\phi_{1I} = \phi^*_{2I} = \frac{(x-x_1)_\mu \bar{\tau}_\mu}{[(x-x_1)^2+\rho_1^2]^{1/2}}\, v \qquad \phi_{1\bar{I}} = \phi^*_{2\bar{I}} = \frac{(x-x_2)_\mu \tau_\mu}{[(x-x_1)^2+\rho_1^2]^{1/2}}\, v \tag{5}$$

where $\eta_{a\mu\nu}$, $\bar{\eta}_{a\mu\nu}$ are the 't Hooft symbols, x_1, x_2, ρ_1, ρ_2 are the location and size of the instanton and anti-instanton respectively, and $\bar{\tau}^\dagger_\mu = \tau_\mu = (\mathrm{i}\tau_i, \mathbb{1})$. The scalar fields are found as solutions to the Euclidean equations of motion $\mathscr{D}^2\phi_1 = 0$, $\mathscr{D}^2\phi^*_2 = 0$, whereas the instanton is found from the self-duality condition. This, however, is not a solution of the Euclidean equation of motion because of the background of the scalar fields. An exact solution does not exist provided the scalars' contribution to the action does not vanish. This can be proven along the line originated by Derrick [5] (Derrick's theorem) by using a simple scaling argument, as will be shown next.

Classically the fermionic fields vanish and the classical Euclidean action (or the energy for static solutions) in D dimensions can be written as

$$S^D_{\mathrm{cl}} = \int \mathrm{d}^D x[\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2}(\mathscr{D}_\mu\phi)^\dagger \mathscr{D}_\mu\phi + V(\phi)]. \tag{6}$$

Here we take for simplicity one scalar field ϕ and $V(\phi)$ is the self-interaction. By scaling the fields

$$A^a_\mu(x) \to \lambda A^a_\mu(\lambda x) \tag{7a}$$

$$\phi(x) \to \phi(\lambda x) \tag{7b}$$

we find

$$S^D_{\mathrm{cl}} = \int \mathrm{d}^D x[\tfrac{1}{4}\lambda^{4-D}F^a_{\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2}\lambda^{2-D}(\mathscr{D}_\mu\phi)^\dagger \mathscr{D}_\mu\phi + \lambda^{-D}V(\phi). \tag{8}$$

The equations of motion are found by requiring the action (or the energy, for static configurations) to be stationary under an arbitrary variation of the fields, and in particular also under the scale transformation (7). Thus S^D_{cl} cannot be stationary with respect to this variation if all the terms in (8) are either increasing or decreasing functions of λ. As a result, solutions can exist in $D=1$ (kinks, in which case the gauge potentials are absent), $D=2$ (vortices) and $D=3$ (monopoles). In $D=4$ there is a classical solution only for pure Yang-Mills theory (instantons), and by including the interaction with the scalar fields the instantons are no longer solutions of the classical equations of motion [5].

As mentioned above an exact minimum of the action can be found in $D=4$ only when the scalar fields' contribution to the action vanishes. Thus, when $v=0$ we have a trivial topology of the scalar fields' configuration space and the instantons alone provide the tunnelling amplitude for $|n\rangle \to |n+1\rangle$. The Pontryagin index of the instantons is the difference between the indices of the vacua $|n\rangle$ and $|n+1\rangle$, as was proven in [6]. When $v \neq 0$ the configuration space of the scalar fields is topologically non-trivial, and the configurations (5) together with the instantons (4) provide the tunnelling between $|n, n\rangle \to |n+1, n+1\rangle$. To see that we calculate the index. The Pontryagin index is given by

$$I_{\mathrm{P}} = \frac{g^2}{32\pi^2}\int \mathrm{d}^4 x F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} \tag{9}$$

where

$$\tilde{F}^a_{\mu\nu}=\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^a_{\rho\sigma}. \tag{10}$$

It is gauge invariant and a total divergence

$$F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}=\partial_\mu K_\mu=\tfrac{1}{2}\varepsilon_{\mu\gamma\rho\sigma}\partial_\mu\,\mathrm{Tr}(A_\nu F_{\rho\sigma}-\tfrac{2}{3}gA_\nu A_\rho A_\sigma). \tag{11}$$

Choosing the gauge $A_0=0$ we find

$$I_\mathrm{P}=n(t=\infty)-n(t=-\infty)=\Delta n \tag{12}$$

where n is the vacuum index of the gauge potentials:

$$n=\frac{1}{24\pi^2}\int_{S_3^\infty}\mathrm{d}\Sigma\;\varepsilon^{ijk}\tfrac{1}{2}\,\mathrm{Tr}(U^{-1}\partial_i UU^{-1}\partial_j UU^{-1}\partial_k U). \tag{13}$$

Here we use the fact that, for $t=\pm\infty$, the field strength vanishes and the gauge potential is a pure gauge. However, at $t=\pm\infty$ we also have $\mathscr{D}_\mu\phi=0$ (ϕ represents either ϕ_1 or ϕ_2^*) or $\phi=U(x)v$, and we may therefore write I_P as

$$I_\mathrm{P}=k(t=\infty)-k(t=-\infty)=\Delta k \tag{14}$$

where k is the vacuum index of the scalar fields as given in [7]. Expressing ϕ in terms of real fields $\phi\equiv\binom{\chi_1+\mathrm{i}\chi_2}{\chi_3+\mathrm{i}\chi_4}$ the vacuum index k is given by [7]

$$k=\frac{1}{12\pi^2}\int_{S_3^\infty}\mathrm{d}\Sigma\;\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma\delta}\chi_\alpha\partial_i\chi_\beta\partial_j\chi_\gamma\partial_k\chi_\delta. \tag{15}$$

Thus I_P is the difference between the vacuum indices both of the scalar fields and the gauge potentials, and we also have $k=n$, as has already been mentioned above. As the field configurations (4) and (5) interpolate between the fields at $t=-\infty$ and $t=\infty$ with $I_\mathrm{P}=1(\Delta n=\Delta k=1)$ we find the tunnelling $|n,\,n\rangle\to|n+1,\,n+1\rangle$ by taking (4) for the instantons and (5) for the scalar fields.

The Euclidean action of these field configurations is finite:

$$S_\mathrm{E}=8\pi^2/g^2+4\pi^2v^2\rho^2 \tag{16}$$

where the first term is the instanton's (anti-instanton's) contribution and the second is the contribution of the scalar fields (5). It results from the kinetic energy and depends on the instanton's size, ρ. Note that $\rho=0$ minimises the action for a given index. This is the case when the scalars' contribution to the action vanishes. We note also that for $v=0$ we have the instanton's action $8\pi^2/g^2$.

We analyse the massive theory in a similar way. We need first to take into account the extra mass term:

$$\mathscr{L}_\mathrm{mass}=-\phi_1^\dagger m^2\phi_1-\phi_2^\dagger m^2\phi_2-\psi_1^\mathrm{T}m\psi_2-\bar{\psi}_1^\mathrm{T}m\bar{\psi}_2. \tag{17}$$

It destroys the flat directions of the potential of the massless theory (ϕ_1 parallel to ϕ_2^*). Thus the vacuum is uniquely defined by $\phi_1=\phi_2=\psi_1=\psi_2=\lambda=0$, $A^a_\mu\frac{1}{2}\tau^a=(\mathrm{i}/g)U^{-1}(x)\partial_\mu U(x)$, and the Pontryagin index labels the gauge potentials only. The scalar fields are topologically trivial. This means that quantum mechanical tunnelling between vacua differing by one unit of topological charge is provided by the instantons or the anti-instantons (4) with $\phi_1=\phi_2=0$. Indeed, if instead one takes the non-trivial configurations of (5) when the mass term is included, the classical action diverges and quantum mechanical tunnelling is completely suppressed†.

† It should be noted that tunnelling in the massive theory was calculated with the configurations (5) in [1, 2] ignoring the infinite action of these configurations. In [3], on the other hand, the configuration $v=0$ in the massive theory was used.

We end this paper with some general remarks about vacuum structure in theories which include scalar fields. Generally, the vacuum is classified by the Pontryagin index characterising the maps $\pi_3(G)$ of the zero-energy gauge potentials. If there is no accidental symmetry and G is broken down to H by the set of minima, M, of the potential of the scalar fields, M can be identified with the coset space G/H. As a result, the set of zero-energy scalar fields define the maps $\pi_3(\mathrm{G/H})$, classified by an index k. A relation between k and the Pontryagin index n, can be found from the exact sequence of maps

$$\ldots \pi_3(\mathrm{H}) \xrightarrow{i} \pi_3(\mathrm{G}) \xrightarrow{j} \pi_3(\mathrm{G/H}) \xrightarrow{k} \pi_2(\mathrm{H}) \xrightarrow{l} \pi_2(\mathrm{G}) \ldots \tag{18}$$

which means that

$$\mathrm{Im}\, i = \mathrm{Ker}\, j \tag{19a}$$

$$\mathrm{Im}\, j = \mathrm{Ker}\, k. \tag{19b}$$

Thus, for example, if H is Abelian (the identity or a product of U(1) factors) $\pi_3(\mathrm{H}) = e$ (the identity) and $\pi_2(\mathrm{H}) = e$†. Therefore $\pi_3(G) \simeq \pi_3(\mathrm{G/H})$ and $k = n$. The example of such a case is $\mathrm{G} = \mathrm{SU}(2)$, ϕ in the fundamental representation ($\mathrm{H} = \mathrm{I}$) or ϕ in the adjoint representation ($\mathrm{H} = \mathrm{U}(1)$). In both cases $k = n$. This was proven in [7] where also the indices of the maps $\pi_3(\mathrm{G/H})$ for both cases were explicitly shown to be equal to n.

In the above, where we analysed massive and massless SQCD, $\mathrm{G/H} = S_3$ in the massless theory ($\mathrm{G} = \mathrm{SU}(2)$, $\mathrm{H} = \mathrm{I}$) and it is one point ($\phi_1 = \phi_2 = 0$) in the massive theory (the group G is unbroken). Therefore $k = n$ (or $k = 0$) in the massless theory and $k = 0$ in the massive theory (which is achieved by taking v, the vacuum expectation value of the scalar fields, to be zero).

To summarise we have pointed out in the above that vacuum structure in theories which include scalar fields (such as SQCD) depends not only on the topology of the gauge group but also on the topology of the set of minima of the scalar potential. In particular, this topological structure depends on whether the theory (SQCD) is massive or massless. As a result, different field configurations induce the tunnelling in the massive and the massless theory ($\phi_1 = \phi_2^* = U(x)v$ for the massless theory and $\phi_1 = \phi_2^* = 0$ for the massive one, accompanied, of course, by the instanton's gauge potentials).

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† This is Cartan's theorem: the second homotopy of all Lie groups is the identity. $\pi_2(\mathrm{H}) = e$ is a particular example.